

Irreversibility, Uncertainty and Investment in the Presence of Sequential Technology Generations

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Abstract

We apply a real options approach to develop a general characterization of the dynamics of the capital accumulation process in the presence of sequential technology generations. In particular, we delineate circumstances under which the present technology represents a compound real option, which incorporates as valuable embedded options the opportunity of successive updating of the technology to superior future technologies with a stochastic pattern of arrival timings. We characterize general circumstances under which the optimal capital stock is decreasing as a function of the expected delay until the new technology arrives.

Keywords: Real options, capital accumulation, sequential technology generations.

JEL Subject Classification: O32 , G30 , D92 , C61.

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1 Introduction

In this paper we consider the emergence of sequential technology generations as a random and exogenous process. The firm can upgrade its current technology to a superior new one with a timing pattern, which is determined by a sequence of exponentially distributed arrival dates. The present model is designed for the analysis of the dynamic investment behavior of firms able to enjoy spillovers from technological innovations, which originate from other firms in the industry or from other sectors of the economy. The present model does not analyze the dynamics of the capital accumulation process for firms actively engaged in R&D activities. Nevertheless the model certainly captures a number of key aspects of the capital accumulation process for a large proportion of sectors in the economy. For example, the technological progress taking place within the framework of the information technology certainly has significant implications for the dynamics of the investment programs of firms in retailing or banking even though these firms do not typically engage in IT-oriented R&D activities. As another example we could think of such types of investments into human capital whereby firms broaden the knowledge base of their employees as an adjustment process in anticipation of future, but still uncertain, technological progress.

In our model technological progress is stochastic, but exogenous, and it is characterized in a very general way. In fact, the model distinguishes four different ways in which adoption of a new technology impacts on the cash flow of the firm. (1) The upgraded technology is revenue enhancing, the underlying source of which could be demand-enhancing quality improvements. (2) The upgraded technology is efficiency-enhancing in the sense that it is cost-reducing, i.e. technological progress reduces the cost of an effective unit of investment. (3) The upgraded technology changes the durability of the product, because it affects the rate at which the capital stock depreciates. (4) The volatility of the capital stock process is specific for each technology generation. Within such a context we apply a real options approach to characterize the process of optimal capital accumulation in a general way¹. Our model delineates how anticipations of future, still uncertain technological progress affect the time profile of firms' investments and, in particular, current investment behavior.

Our study explores how the presence of future uncertain technological progress affects the real option value associated with investments within the framework of a currently existing technology.

¹In a mathematical sense, our model has similarities with the analysis applied in section 6 of Baldursson and Karatzas 1997.

These investments generate a capital stock, which, of course, depreciates over time, but which can potentially also be more efficiently utilized within the framework of superior future technology versions. Thus, current investments represent a compound real option as these investments are linked to an opportunity of exploiting future, still uncertain technology improvements.

We analyze the dynamics of the firm's optimal capital stock. In a mathematical sense we investigate the capital stock threshold of a compound real option. In particular, we address the following questions. How does the presence of future technological progress impact on the firm's optimal capital stock with respect to the present technology? More precisely, what is the relationship between the capital stock threshold of the present technology and the expected delay until the improved technology arrives? What is the relationship between the optimal capital stock of the present technology and that associated with the updated technology? Our analysis of these questions offers important insights for the overall understanding of the optimal pattern of capital accumulation in the presence of technological progress.

In general, the relationship between the investment threshold levels depend on all the parameters involved: (1) the revenue-enhancing effect, (2) the cost-reducing effect, (3) the durability-enhancing effect and (4) the volatility effect. In addition, the expected delay in the arrival of the new technology or the technological uncertainty, i.e. the inverse hazard rate, seems to be particularly important. We find that the impact of technological progress on the investment threshold depends on the way in which adoption of a new technology affects the cash flow or the costs of the firm. For example, with constant rates of depreciation and volatility we find that technological improvements in the form of cost reductions or revenue increases will induce a monotonic sequence of continuously increasing capital stock thresholds. Thus, under these circumstances the capital stock thresholds always increase as we proceed towards more advanced technology generations.

Our analysis establishes that the impact of technological progress on the capital accumulation process is not restricted to the modifications of the investment thresholds. In fact, we characterize how anticipated technological progress affects the probability of increasing the investment level from that associated with the present technology towards the higher level associated with the more productive future technology as a function of the prevailing state in the present technology. Namely, as the capital stock hits the threshold for investing with the present technology the firm also has an incentive to increase its capital stock so as to adjust its operation to the improved technology.

A number of influential research contributions has previously analyzed various aspects of

optimal sequential investment behavior for firms (cf. Arrow 1968, Baldwin 1982, Nickell 1974). The real options approach has extended this analysis making it possible to characterize optimal investment in the presence of irreversibility, uncertainty, and multi-stage projects. Dixit and Pindyck (1994) determine the real option values of investment opportunities available at each stage of a multi-stage project and they characterize critical thresholds (with respect to the underlying state variable) that trigger the investments. Dutta (1994) applies dynamic programming to characterize the optimal allocation of an R&D budget between several interrelated stages of a project with the feature that completion of an intermediate step might itself be profitable. His analysis establishes that the optimal path of R&D expenditures is decreasing in the sense that a larger share of the budget should be allocated to earlier stages of the project.

Our study differs from those mentioned above in several important ways. Our analysis focuses on the impact of future, but still uncertain, generations of technology improvements on the option value of adopting an incumbent technology as well as on the investment volumes. Especially, we delineate how the compound real option value associated with the acquisition of the incumbent technology will exhibit crucial dependence on market uncertainty, future technological uncertainty, the depreciation of the capital stock, the generation-specific volatility of the stochastic process associated with a particular technology as well as the interaction between these characteristics. In these respects the present study has certain features in common with Alvarez and Stenbacka (2001), which calculates the optimal timing of when to adopt an incumbent technology, incorporating as an embedded option a technologically uncertain prospect of updating to superior future technology generations. With attention restricted to technology-specific risk aspects this study is also related to Alvarez and Stenbacka (2004). Relative to these contributions, with their focus restricted to the optimal threshold of when to simply adopt an available technology, the present study represents an essential generalization by focusing on the *optimal capital accumulation* process incorporating the intertemporal delineation of the *investment volumes*. In addition, our present approach offers a four-dimensional representation of technological progress.

Our analysis proceeds as follows. In Section 2, we present our general stochastic model in a setting with two technology generations and we characterize the optimal capital accumulation process as a function of the underlying market uncertainty, the depreciation rate as well as the generic technological uncertainty. The general model is illustrated for particular functional forms at the end of the section. In Section 3 the general model is extended to a horizon with an arbitrary number of sequential technology generations. Finally, we offer some concluding

comments in Section 4.

2 The Model

We assume that the dynamics of the capital stock evolve according to the process $\{K_t^I; t \in \mathbb{R}_+\}$ defined on a complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F})$ satisfying the usual conditions and described on \mathbb{R}_+ by the (Itô-) stochastic differential equation

$$dK_t^I = -\delta_t K_t^I dt + \sigma_t K_t^I dW_t + dI_t, \quad K_0^I = k, \quad (2.1)$$

where $\delta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ denotes the percentage depreciation rate of the capital stock, $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ denotes the volatility of the capital stock process, and I_t is an *admissible irreversible investment strategy*. We call an irreversible investment strategy I_t admissible if it is non-negative, non-decreasing, right-continuous and adapted, and denote the set of admissible policies as Λ . In order to capture the impact of technological progress on the capital stock dynamics, we assume that the depreciation rate process and the volatility process are defined as

$$(\delta_t, \sigma_t) = \begin{cases} (\delta_2, \sigma_2) & t \geq \tau \\ (\delta_1, \sigma_1) & t < \tau, \end{cases} \quad (2.2)$$

where $\delta_1, \delta_2, \sigma_1$ and σ_2 are known non-negative real-valued constants. The parameter τ , denoting the random arrival date of the upgraded production technology, is assumed to be exponentially distributed with parameter $\lambda > 0$. Consequently, we observe that the dynamic process of capital accumulation is subject to stochastic depreciation and volatility, since both the depreciation and volatility coefficient are determined by the random arrival date τ .

In order to characterize the impact of the technological improvement on the net revenues of the firm, we assume that the short-run revenue flow of the firm is

$$\pi(t, k) = \begin{cases} \pi_1(k) & t < \tau \\ \pi_2(k) & t \geq \tau, \end{cases} \quad (2.3)$$

where τ denotes the previously defined arrival date of the upgraded production technology and the mappings $\pi_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$, $i = 1, 2$ are assumed to be continuously differentiable, monotonically increasing, strictly concave, and to satisfy the standard *Inada conditions* $\lim_{k \downarrow 0} \pi_i'(k) = \infty$, $\lim_{k \rightarrow \infty} \pi_i'(k) = 0$, $\lim_{k \downarrow 0} \pi_i(k) = 0$, and $\lim_{k \rightarrow \infty} \pi_i(k) = \infty$ for $i = 1, 2$. We call the upgraded technology *revenue enhancing* whenever the revenues associated with the upgraded technology

dominate relative to those of the incumbent technology, that is, whenever $\pi_2(k) > \pi_1(k)$ for all $k \in \mathbb{R}_+$.

Given the definitions above, we now plan to determine the irreversible investment policy $I_t \in \Lambda$ for which the maximum

$$V(t, k) = \sup_{I \in \Lambda} \mathbf{E}_{(t, k)} \int_t^\infty e^{-rs} [\pi(s, K_s^I) ds - q_s dI_s], \quad (t, k) \in \mathbb{R}_+^2 \quad (2.4)$$

is attained. In (2.4) $r > 0$ denotes the risk free discount rate and

$$q_t = \begin{cases} q_1 & t < \tau \\ q_2 & t \geq \tau, \end{cases}$$

denotes the unit cost of investment. In other words, the objective of the firm is to determine an optimal irreversible investment policy maximizing the expected cumulative present value of the future net cash flows from the present up to an arbitrarily distant future. We call the upgraded technology *cost-reducing* if the unit cost of investment is expected to decrease at the arrival date of the new technology, that is, if $q_1 > q_2$.

We now denote as K_t the capital stock process K_t^I in the absence of investment (that is, when the capital stock is left depreciating at the expected rate δ_t). Given the stochastic differential equation (2.1), the capital stock process K_t constitutes a coupled diffusion process

$$K_t = \begin{cases} K_t^1, & t < \tau \\ K_t^2, & t \geq \tau, \end{cases} \quad (2.5)$$

where τ is the exponentially distributed random coupling date, and K_t^1 and K_t^2 denote the capital accumulation process associated with the incumbent and the upgraded technology, respectively. These processes are defined on \mathbb{R}_+ by the stochastic differential equation

$$dK_t^i = -\delta_i K_t^i dt + \sigma_i K_t^i dW_t, \quad i = 1, 2.$$

2.1 The Optimal Capital Accumulation Policy with the Updated Technology

The stochastic optimal investment problem (2.4) has to be solved, as typically, in two separate steps. We first consider the optimal irreversible investment policy and the value of the firm after the upgraded technology has arrived and been adopted, that is, when the condition $t \geq \tau$ is satisfied. Having solved these quantities we then invoke the principle of optimality and solve

the value prior to the arrival of the upgraded technology conditional on the optimal investment policy and its value in the presence of this upgraded technology.

We first observe that the optimal investment problem (2.4) is time-invariant during the phase after the arrival of the updated technology. This means that the value of the optimal investment policy is independent of time on the set $[\tau, \infty)$. More precisely, if $t \geq \tau$ it holds that

$$V(t, k) = V_2(k) = \sup_{I \in \Lambda} \mathbf{E}_k \int_0^\infty e^{-rs} [\pi_2(K_s^I) ds - q_2 dI_s], \quad (2.6)$$

which is an ordinary problem in singular stochastic control. This can be solved by standard techniques (cf. Alvarez 1999). In (2.6) the operator \mathbf{E}_k captures that the expectation is formed at the current date. Applying the generalized Itô theorem to the mapping $(t, k) \mapsto e^{-rt} q_2 k$ implies that for all $(t, k) \in \mathbb{R}_+^2$ we have the inequality

$$V_2(k) \leq q_2 k + \sup_{I \in \Lambda} \mathbf{E}_k \int_0^\infty e^{-rs} [\pi_2(K_s^I) - (r + \delta_2) q_2 K_s^I] ds, \quad (2.7)$$

where the term $(r + \delta_2) q_2 k$ is the familiar *user cost of capital* (cf. Jorgenson 1963). Before proceeding in our analysis, we first define the mapping $\theta_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$\theta_2(k) = \pi_2(k) - (r + \delta_2) q_2 k.$$

The strict concavity of the mapping $\pi_2(k)$ and the Inada-conditions guarantee that the mapping $\theta_2(k)$ attains a unique global maximum at the point $\tilde{k}_2 = \operatorname{argmax}\{\theta_2(k)\}$ satisfying the necessary first order condition $\theta_2'(\tilde{k}_2) = 0$. Moreover, since the mappings $\theta_2(k)$ and $\pi_2(k)$ have finite expected cumulative present values by the concavity, non-negativity and continuous differentiability of $\pi_2(k)$ (i.e. the *absence of speculative bubbles condition* is met), we also observe that

$$(R_r^2 \theta_2)(k) = (R_r^2 \pi_2)(k) - q_2 k,$$

where the functional $(R_r^2 f) : \mathbb{R}_+ \mapsto \mathbb{R}$ is defined for an arbitrary mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}$ with finite expected cumulative present value as

$$(R_r^2 f)(k) = \mathbf{E}_k \int_0^\infty e^{-rs} f(K_s^2) ds.$$

Instead of tackling the stochastic control problem (2.6) directly, we follow the approach introduced in Alvarez (1999), and consider first the associated ordinary non-linear programming problem

$$\sup_{k \in \mathbb{R}_+} \{k^{1-\varphi_2} (R_r^2 \theta_2)'(k)\}, \quad (2.8)$$

where $\varphi_2 < 0$ denotes the negative root of the characteristic equation $\sigma_2^2 a(a-1)/2 - \delta_2 a - r = 0$.

Our first auxiliary result is now summarized in the following

Lemma 2.1. *There is a unique threshold*

$$k_2^* = \operatorname{argmax} \{k^{1-\varphi_2} (R_r^2 \theta_2)'(k)\} \in (0, \tilde{k}_2)$$

satisfying the ordinary first order condition $k_2^* (R_r^2 \theta_2)''(k_2^*) = (\varphi_2 - 1) (R_r^2 \theta_2)'(k_2^*)$ which can be re-expressed as

$$\int_{k_2^*}^{\infty} y^{-\psi_2} \theta_2'(y) dy = 0, \quad (2.9)$$

where $\psi_2 > 1$ denotes the positive root of the characteristic equation $\sigma_2^2 a(a-1)/2 - \delta_2 a - r = 0$.

Proof. See Appendix A. □

In light of Lemma 2.1, we can now prove the following

Theorem 2.2. *The optimal capital stock threshold $k_2^* \in (0, \tilde{k}_2)$ associated with the upgraded technology is the unique root of the ordinary first order condition (2.9). The associated value of the firm*

$$V_2(k) = \begin{cases} q_2 k + (R_r^2 \theta_2)(k) - \frac{1}{\varphi_2} k_2^* (R_r^2 \theta_2)'(k_2^*) (k/k_2^*)^{\varphi_2} & k > k_2^* \\ q_2 k + \theta_2(k_2^*)/r & k \leq k_2^* \end{cases} \quad (2.10)$$

is twice continuously differentiable, concave, and satisfies the inequality $0 \leq V_2'(k) \leq q_2$. Moreover, $V_2'(k_2^*) = q_2$, $V_2''(k_2^*) = 0$, and $\lim_{k \rightarrow \infty} V_2'(k) = 0$.

Proof. See Appendix B. □

Theorem 2.2 demonstrates that under the conditions of our study a unique optimal capital accumulation policy characterized by a single exercise threshold exists. Under the optimal policy, the maximal attainable expected cumulative present value of the future cash flows reads as in (2.10). Interestingly, we observe that (2.10) implies that the optimal policy with the upgraded technology generates an *excess return* according to

$$V_2(k) - q_2 k = \begin{cases} (R_r^2 \theta_2)(k) - \frac{1}{\varphi_2} k_2^* (R_r^2 \theta_2)'(k_2^*) (k/k_2^*)^{\varphi_2} & k > k_2^* \\ \theta_2(k_2^*)/r & k \leq k_2^*. \end{cases}$$

It is worth noticing that the value (2.10) associated with the optimal capital accumulation rule implies that the standard balance equation $V_2(k_2^*) = V_2(k) + q_2(k_2^* - k)$ holds for all $k \leq k_2^*$.

This balance equation states that at the optimum the value of the project has to be equal to its full costs measured by the sum of the lost option value $V_2(k)$ (indirect costs) and the direct investment costs $q_2(k_2^* - k)$.

The main comparative static properties of the optimal capital accumulation policy and its associated value are now summarized in the following.

Theorem 2.3. (*Comparative static properties*) *The value function $V_2(k)$ satisfies the conditions $\partial V_2(k)/\partial\sigma_2 < 0$, $\partial V_2(k)/\partial r < 0$, $\partial V_2(k)/\partial\delta_2 < 0$, and $\partial V_2(k)/\partial q_2 < 0$. The optimal capital stock threshold satisfies the conditions $\partial k_2^*/\partial\sigma_2 < 0$, $\partial k_2^*/\partial r < 0$, $\partial k_2^*/\partial\delta_2 < 0$, and $\partial k_2^*/\partial q_2 < 0$. Moreover, the marginal value $V_2'(k)$ satisfies the conditions $\partial V_2'(k)/\partial q_2 > 0$, $\partial V_2'(k)/\partial r < 0$ and $\partial V_2'(k)/\partial\delta_2 < 0$.*

Proof. See Appendix C. □

Theorem 2.3 demonstrates that increased volatility (discounting, depreciation, unit cost of investment), *ceteris paribus*, decreases the value and postpones the rational exercise of the sequential investment opportunity by decreasing the optimal investment threshold k_2^* . This finding can be explained as follows. Although increased volatility decreases the lost option value $V_2(k)$ it simultaneously decreases the value $V_2(k_2^*)$ as well. Since the latter effect dominates the former, we find from the balance equation that increased volatility unambiguously decreases the optimal capital stock threshold k_2^* .

As is well known from the neoclassical theory of investment, the marginal value $V_2'(k)$ can be interpreted as *Tobin's (marginal) q* of investment. In light of Lemma 2.1 this marginal value reads as

$$V_2'(k) = (R_r^2 \pi_2)'(k) - k^{\varphi_2 - 1} \sup_{y \leq k} [y^{1 - \varphi_2} (R_r^2 \theta_2)'(y)]. \quad (2.11)$$

Thus, our results indicate that the optimal policy does not only maximize the expected cumulative present value of the future cash flows, but it also simultaneously maximizes the rate at which this value grows as can be observed from the representation of Tobin's q in terms of an associated ordinary non-linear programming problem stated in (2.11). As in standard neoclassical models of investment, the term $(R_r^2 \pi_2)'(k)$ measures the expected cumulative present value of the future marginal revenue productivity of the operating capital stock. The latter term of (2.11) measures the early exercise premium associated with exploiting the investment opportunity prior to its expiration. Naturally, the growth rate of the excess returns associated with the optimal capital

accumulation policy now reads as

$$V_2'(k) - q_2 = (R_r^2 \theta_2)'(k) - k^{\varphi_2 - 1} \sup_{y \leq k} [y^{1 - \varphi_2} (R_r^2 \theta_2)'(y)].$$

Given (2.12) and the identity $(R_r^2 \theta_2)'(k) = (R_r^2 \pi_2)'(k) - q_2$, we find that the marginal value can be re-expressed as

$$V_2'(k) = (R_r^2 \pi_2)'(k) - \sup_{\tau} \mathbf{E}_k \left[e^{-(r+\delta)\tau} ((R_r \pi)'(\hat{K}_\tau^2) - q_2) \right], \quad (2.12)$$

where the process \hat{K}_t^2 evolves according to the dynamics described by the stochastic differential equation

$$d\hat{K}_t^2 = (\sigma_2^2 - \delta_2) \hat{K}_t^2 dt + \sigma_2 \hat{K}_t^2 dW_t, \quad \hat{K}_0^2 = k. \quad (2.13)$$

The representation (2.12) shows how the original investment problem (2.6) is related to the modern real option models of irreversible investment and, especially, to models considering optimal exit and the valuation of the associated put options. As (2.12) clearly indicates the marginal value and, thus, *Tobin's marginal q* can be seen to be generated by an associated optimal timing problem (*a perpetual American forward contract written on a dividend-paying asset*). Consequently, we find that neoclassical problems of investment can be tackled either directly by considering the optimal investment problem and, therefore, the valuation of the firm or indirectly by determining the optimal marginal value of the operating capital stock.

It is worth emphasizing that in the present case the capital stock dynamics associated with the updated technology tend towards a random stationary steady state distributed according to a Pareto-distribution $p(k)$ with density

$$p'(k) = (\psi_2 + \varphi_2) k_2^* \psi_2 + \varphi_2 k^{-\psi_2 - \varphi_2 - 1}$$

(compare with Merton 1975). Hence, the expected long-run steady state of the capital stock is

$$\lim_{t \rightarrow \infty} \mathbf{E}_k [K_t^2] = \left(1 + \frac{\sigma_2^2}{2\delta_2} \right) k_2^*.$$

2.2 The Optimal Capital Accumulation Policy with the Incumbent Technology

The memoryless-property of the exponential distribution and the strong Markov property of linear diffusions implies that the time-invariance of the optimal investment problem can be extended to the phase prior to the arrival of the new upgraded technology as well. More precisely,

since the conditional distribution of the arrival date is exponential, we find that for any $t < \tau$ the investment problem reads as

$$V(t, k) = V_1(k) = \sup_{I \in \Lambda} \mathbf{E}_k \left[\int_0^\tau e^{-rs} [\pi_1(K_s^I) ds - q_1 dI_s] + e^{-r\tau} V_2(K_\tau^I) \right]. \quad (2.14)$$

Applying again the generalized Itô theorem to the mapping $(t, k) \mapsto e^{-rt} q_1 k$ and changing the order of integration results in the inequality

$$V_1(k) \leq q_1 k + \sup_{I \in \Lambda} \mathbf{E}_k \int_0^\infty e^{-(r+\lambda)s} \theta_1(K_s^I) ds, \quad (2.15)$$

where

$$\theta_1(k) = \pi_1(k) - (r + \delta_1) q_1 k + \lambda(V_2(k) - q_1 k).$$

There is one major difference caused by the uncertainty associated with the timing of the regime switch. For that reason the flow $\theta_1(k)$ differs from that of $\theta_2(k)$ in a qualitative sense by the term $\lambda(V_2(k) - q_1 k)$, which measures the true net gain of switching from the present technology to the upgraded technology.

It is clear that the concavity of the value $V_2(k)$ and the strict concavity of the mapping $\pi_1(k)$ imply that the mapping $\theta_1(k)$ is strictly concave. Moreover, $\theta_1(k)$ satisfies the limiting conditions

$$\lim_{k \downarrow 0} \theta_1(k) = \frac{\lambda}{r} \theta_2(k_2^*) > 0,$$

$\lim_{k \downarrow 0} \theta_1'(k) = \infty$ and $\lim_{k \rightarrow \infty} \theta_1'(k) = -(r + \delta_1 + \lambda) q_1 < 0$. Thus, the strict concavity of the mapping $\theta_1(k)$ imply that there is a unique threshold \tilde{k}_1 satisfying the ordinary first order condition $\theta_1'(\tilde{k}_1) = 0$. Since $V_2'(k) \geq 0$ we find that

$$\theta_1'(k) = \pi_1'(k) - (r + \delta_1) q_1 + \lambda(V_2'(k) - q_1) \geq \pi_1'(k) - (r + \delta_1 + \lambda) q_1,$$

which implies that \tilde{k}_1 is above the unique threshold \hat{y} satisfying the first order condition $\pi_1'(\hat{y}) = (r + \delta_1 + \lambda) q_1$. In particular, if we consider a cost-reducing upgrade we can conclude that

$$\theta_1'(k) = \pi_1'(k) - (r + \delta_1) q_1 + \lambda(V_2'(k) - q_1) \leq \pi_1'(k) - (r + \delta_1) q_1.$$

Consequently, the threshold \tilde{k}_1 at which $\theta_1(k)$ is maximized is below the threshold \hat{y} satisfying the ordinary first order condition $\pi_1'(\hat{y}) = (r + \delta_1) q_1$.

Given these observations, we again consider the ordinary non-linear-programming problem

$$\sup_{k \in \mathbb{R}_+} \{k^{1-\varphi_1} (R_{r+\lambda}^1 \theta_1)'(k)\}, \quad (2.16)$$

where $\varphi_1 < 0$ denotes the negative root of the characteristic equation $\sigma_1^2 a(a-1)/2 - \delta_1 a - (r+\lambda) = 0$ and

$$(R_{r+\lambda}^1 \theta_1)(k) = \mathbf{E}_k \int_0^\infty e^{-(r+\lambda)s} \theta_1(K_s^1) ds$$

denotes the expected cumulative present value of the revenue flow $\theta_1(k)$. Our first auxiliary result is now summarized in the following.

Lemma 2.4. *There is a unique threshold*

$$k_1^*(\lambda) = \operatorname{argmax} \{k^{1-\varphi_1} (R_{r+\lambda}^1 \theta_1)'(k)\} \in (0, \tilde{k}_1)$$

satisfying the ordinary first order condition $k_1^*(\lambda) (R_{r+\lambda}^1 \theta_1)''(k_1^*(\lambda)) = (\varphi_1 - 1) (R_{r+\lambda}^1 \theta_1)'(k_1^*(\lambda))$ which can be re-expressed as

$$\int_{k_1^*(\lambda)}^\infty y^{-\psi_1} \theta_1'(y) dy = 0, \quad (2.17)$$

where $\psi_1 > 1$ denotes the positive root of the characteristic equation $\sigma_1^2 a(a-1)/2 - \delta_1 a - (r+\lambda) = 0$.

Proof. The proof of this claim is completely analogous to the proof of Lemma 2.1. \square

Although Lemma 2.4 demonstrates that the ordinary non-linear programming problem (2.16) attains a global minimum at a uniquely determined threshold, it does not characterize the precise location of that threshold, especially not in comparison with the optimal capital stock threshold k_2^* . In light of Lemma 2.4 we can now prove the following

Theorem 2.5. *Prior to the arrival of the new upgraded technology the optimal capital stock threshold $k_1^*(\lambda) \in (0, \tilde{k}_1)$ is the unique root of the ordinary first order condition (2.17). The associated value of the firm*

$$V_1(k) = \begin{cases} q_1 k + (R_{r+\lambda}^1 \theta_1)(k) - \frac{1}{\varphi_1} k_1^*(\lambda) (R_{r+\lambda}^1 \theta_1)'(k_1^*(\lambda)) (k/k_1^*(\lambda))^{\varphi_1} & k > k_1^*(\lambda) \\ q_1 k + \theta_1(k_1^*(\lambda))/(r+\lambda) & k \leq k_1^*(\lambda) \end{cases} \quad (2.18)$$

is twice continuously differentiable, concave, and satisfies the inequality $0 \leq V_1'(k) \leq q_1$. Moreover, $V_1'(k_1^*(\lambda)) = q_1$, $V_1''(k_1^*(\lambda)) = 0$, and $\lim_{k \rightarrow \infty} V_1'(k) = 0$.

Proof. The proof is analogous to the proof of Theorem 2.2. \square

Theorem 2.5 extends the results of Theorem 2.2 to the phase of the incumbent technology. An important implication of (2.18) is that during this phase the excess return associated with

the optimal capital accumulation policy reads as

$$V_1(k) - q_1 k = \begin{cases} (R_{r+\lambda}^1 \theta_1)(k) - \frac{1}{\varphi_1} k_1^*(\lambda) (R_{r+\lambda}^1 \theta_1)'(k_1^*(\lambda)) (k/k_1^*(\lambda))^{\varphi_1} & k > k_1^*(\lambda) \\ \theta_1(k_1^*(\lambda)) / (r + \lambda) & k \leq k_1^*(\lambda). \end{cases}$$

Since this expression depends on the cash flow $\theta_1(k)$ and the arrival intensity λ we can conclude that the excess return associated with the incumbent technology incorporates the value and arrival uncertainty associated with the embedded upgrading opportunity.

Along the lines of our findings on the optimal capital accumulation policy in the presence of the upgraded technology we again find that the value (2.18) associated with the optimal capital accumulation rule implies that the standard balance equation $V_1(k_1^*(\lambda)) = V_1(k) + q_1(k_1^*(\lambda) - k)$ holds for all $k \leq k_1^*(\lambda)$. Consequently, at the optimum the value of the project $V_1(k_1^*(\lambda))$ can be decomposed into the lost option value $V_1(k)$ (indirect costs) and the direct investment costs $q_1(k_1^*(\lambda) - k)$.

A straightforward implication of Theorem 2.5, Lemma 2.1, and Lemma 2.4 characterizing the impact of technological change on the optimal capital accumulation path is now summarized in the following.

Corollary 2.6. *If*

$$\int_{k_2^*}^{\infty} y^{-\psi_1} \theta_1'(y) dy \stackrel{\leq}{\geq} 0,$$

then $k_1^(\lambda) \stackrel{\leq}{\geq} k_2^*$. Consequently, the optimal capital stock decision is neutral with respect to technological change whenever*

$$\int_{k_2^*}^{\infty} y^{-\psi_1} \theta_1'(y) dy = 0. \quad (2.19)$$

Corollary 2.6 presents a sufficient condition for the presence of subsequent technology generations to reduce (increase) the capital stock threshold of the present technology. In particular, the inequality

$$\int_{k_2^*}^{\infty} y^{-\psi_1} \theta_1'(y) dy \leq 0, \quad (2.20)$$

is a sufficient condition under which the presence of uncertain future technological progress will speed up the investment process associated with the presently available technology in relationship with a static world where no technological development could be foreseen. Thus, under inequality (2.20) the present technology represents a compound real option, which incorporates as a valuable embedded real option the opportunity of successive updating of this technology to a superior future generation with a stochastic pattern of arrival timings.

Under particular parameter configurations described by the integral equation (2.19) the presence of future technology generations add no embedded option value to the present technology. This captures a situation where the successive technology generations represent an independent sequence of real options in the sense that the potential emergence of future technologies does not generate any embedded option value.

The arrival intensity $1/\lambda$ of the new technology denotes the expected time until the incumbent technology can be updated. It can immediately be seen that

$$\lim_{\lambda \downarrow 0} V_1(k) = \check{V}_1(k), \quad (2.21)$$

$$\lim_{\lambda \downarrow 0} k_1^*(\lambda) = k^*(0), \quad (2.22)$$

where $\check{V}_1(k)$ denotes the value of the optimal policy in the absence of a future upgrading opportunity and $k_1(0)$ denotes the corresponding optimal capital stock threshold. In general, the relationship between the capital stock thresholds of the two subsequent technologies is determined by the arrival intensity to an essential extent. In fact, the shorter is the expected time until the technology can be updated the larger is the value of the embedded option relative to a situation where the technology cannot be updated. The larger is the value of the embedded option the stronger is the link between the subsequent technologies and the stronger is the impact of technological progress. Similarly, even though waiting is typically valuable also in the presence of technological progress, it is less valuable the stronger is the link between the two subsequent technologies. This naturally implies that the arrival intensity is an important factor characterizing the optimal capital accumulation path. Unfortunately, deriving simple parametric conditions under which the sign of this relationship could be unambiguously determined is difficult, because both the cash flow $\theta_1(k)$ and the discount rate at which the future cash flows are discounted depend on the arrival intensity λ . However, we can establish the following interesting result characterizing the sensitivity of the optimal capital accumulation threshold to changes in the arrival intensity.

Theorem 2.7. *With the incumbent technology the optimal capital stock threshold $k_1^*(\lambda)$ satisfies the condition*

$$k_1^{*'}(\lambda) = \frac{V_1(k_1^*(\lambda)) - V_2(k_1^*(\lambda))}{\theta_1'(k_1^*(\lambda))} = \frac{\pi_1(k_1^*(\lambda)) - \delta q_1 k_1^*(\lambda) - r V_2(k_1^*(\lambda))}{(r + \lambda)\theta_1'(k_1^*(\lambda))}.$$

Therefore, $k_1^{*'}(\lambda) \geq 0$ whenever $V_1(k_1^*(\lambda)) \geq V_2(k_1^*(\lambda))$. Especially, if $k_1^*(\lambda) < k_2^*$ then

$$k_1^{*'}(\lambda) \leq \frac{\pi_1(k_1^*(\lambda)) - \pi_2(k_1^*(\lambda)) - \delta(q_1 - q_2)k_1^*(\lambda)}{(r + \lambda)\theta_1'(k_1^*(\lambda))}.$$

Proof. See Appendix D. □

Consequently, from Theorem 2.7 we can conclude that a shorter expected delay until a value-increasing technology is available induces a lower optimal capital stock. Intuitively, this captures the idea that the firm has a stronger incentive to slow down its investment under the present technology and wait for the arrival of the improved technology the shorter is the expected delay until its arrival.

As in the phase after the arrival of the upgraded technology, the marginal value of the firm has an interesting real option interpretation in terms of an associated optimal exit problem subject to embedded options represented by the arrival of the new technology (cf. Alvarez and Stenbacka 2001). More precisely, since

$$V_1'(k) = q_1 + (R_{r+\lambda}^1 \theta_1)'(k) - k^{\varphi_1-1} \sup_{y \leq k} [y^{1-\varphi_1} (R_{r+\lambda}^1 \theta_1)'(y)], \quad (2.23)$$

we again find that the marginal value can be re-expressed as

$$V_1'(k) = q_1 + (R_{r+\lambda}^1 \theta_1)'(k) - \sup_{\tau} \mathbf{E}_k \left[e^{-(r+\lambda+\delta_1)\tau} (R_{r+\lambda}^1 \theta_1)'(\hat{K}_\tau^1) \right], \quad (2.24)$$

where the process \hat{K}_t^1 evolves according to the dynamics described by the stochastic differential equation

$$d\hat{K}_t^1 = (\sigma_1^2 - \delta_1) \hat{K}_t^1 dt + \sigma_1 \hat{K}_t^1 dW_t, \quad \hat{K}_0^1 = k. \quad (2.25)$$

Thus, in light of our analysis the approach to investment theory based on Tobin's marginal q seems to be justified also in the presence of random technological progress.

2.3 Explicit Illustration

In order to illustrate our results explicitly, we assume that the revenue flows are of the standard exponential form $\pi_i(k) = a_i k^b$, where $b \in (0, 1)$ and $a_i \in \mathbb{R}_+$, $i = 1, 2$, are known exogenously determined constants. We will assume that the upgraded technology is revenue enhancing and, therefore, that $a_2 > a_1$. Consequently, we find that $\theta_2(k) = a_2 k^b - (r + \delta_2) q_2 k$ and, therefore, that

$$\tilde{k}_2 = \left(\frac{a_2 b}{(r + \delta_2) q_2} \right)^{1/(1-b)}.$$

Moreover, taking expectations results in

$$(R_r^2 \theta_2)(k) = \frac{a_2 k^b}{r + m_2(b)} - q_2 k,$$

where $m_i(b) = b\delta_i + \sigma_i^2 b(1-b)/2$ denotes the percentage growth rate of the revenue process. As was established in Lemma 2.1, the optimal capital stock threshold k_2^* is the unique root of the optimality condition $k_2^*(R_r^2\theta_2)''(k_2^*) = (\varphi_2 - 1)(R_r^2\theta_2)'(k_2^*)$ implying that

$$k_2^* = \left(\frac{(\psi_2 - 1)}{(\psi_2 - b)} \frac{a_2 b}{(r + \delta_2) q_2} \right)^{1/(1-b)} = \left(1 - \frac{1-b}{\psi_2 - b} \right)^{1/(1-b)} \tilde{k}_2 < \tilde{k}_2.$$

The value of the firm then reads as

$$V_2(k) = \begin{cases} \frac{a_2 k^b}{r + m_2(b)} - \frac{(1-b)q_2}{\varphi_2(b-\varphi_2)} k_2^{*1-\varphi_2} k^{\varphi_2} & k > k_2^* \\ q_2 k + \left(1 - \frac{1}{\varphi_2}\right) \left(\frac{1-b}{b}\right) q_2 k_2^* & k \leq k_2^*. \end{cases}$$

The optimal policy prior to the arrival of the upgraded technology can now be characterized by making use of $V_2(k)$ and the optimal threshold k_2^* associated with the future upgraded technology. In accordance with the findings of Corollary 2.6, we can now establish the following

Proposition 2.8. $k_1^*(\lambda) \leq k_2^*$ whenever

$$\frac{(\psi_2 - b)(r + \delta_2)}{(\psi_1 - b)(\psi_2 - 1)} \left[\frac{a_1}{a_2} + \frac{\lambda}{(r + m_2(b))} \right] \leq \frac{(r + \delta_1 + \lambda)q_1}{(\psi_1 - 1)q_2} + \frac{\lambda(1-b)}{(b - \varphi_2)(\psi_1 - \varphi_2)}.$$

Proof. See Appendix E. □

Unfortunately, the optimality condition from which the optimal capital stock threshold associated with the incumbent technology has to be determined is non-linear and, therefore, it typically cannot be solved explicitly. However, for technologies exhibiting constant rates of depreciation ($\delta = \delta_1 = \delta_2$) and volatility ($\sigma = \sigma_1 = \sigma_2$) the capital stock thresholds can be explicitly characterized. Under these circumstances it directly follows that $k_1^*(\lambda) \leq k_2^*$ whenever the condition

$$bk^{b-1}(a_1 - a_2) \leq (r + \delta + \lambda)(q_1 - q_2) \tag{2.26}$$

holds for all $k \in \mathbb{R}_+$. Interestingly, condition (2.26) is always satisfied by technology improvements, which either reduce costs while leaving the productivity unchanged or increase productivity while leaving the costs unchanged. Thus, under these circumstances the capital stock threshold is higher for the updated, and more favorable, technology.

We are next interested in exploring how the capital stock threshold of the present technology is affected by the expected delay until the improved technology arrives. We illustrate this by presenting an example.

We exhibit the impact of a cost-reducing technology improvement on the optimal investment strategy as a function of the arrival intensity λ in Figure 1 under the assumptions that

$a_1 = a_2 = 1, \delta_1 = \delta_2 = 0.15, \sigma_1 = \sigma_2 = 0.1, r = 0.03, b = 0.75, q_1 = 1,$ and $q_2 = 0.95$. This parameter configuration captures a fairly moderate cost-reducing technology improvement. Through explicit calculations it can be verified that the optimal capital stock with the updated technology is $k_2^* = 358.043$, whereas the present technology is associated with an optimal capital stock $k_1^*(0) = 291.63$ in the absence of any upgrading opportunity (i.e. formally, with an infinitely long expected delay until the arrival of the new technology). As graphically illustrated in Figure 1, a shorter expected delay until the improved technology is available induces a lower optimal capital stock. This example illustrates the following general lesson. A shorter expected delay until the improved technology arrives induces the firm to slow down its investment activities under the present technology, because with a shorter expected delay it is more profitable to postpone the investments until the improved technology has arrived. Thus, Figure 1 serves as a graphical confirmation of Theorem 2.7 in the case of a cost-reducing new technology.

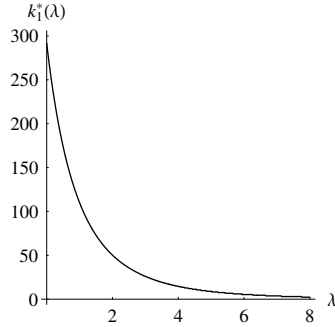


Figure 1: The optimal capital stock threshold $k_1^*(\lambda)$ for a cost-reducing technology improvement

The presence of subsequent technology generations with a random arrival has important effects on the intertemporal investment pattern. In particular, uncertainty regarding the delay until the updated technology generation is available might easily induce interesting "overshooting" effects. We illustrate this with the following example. Consider the parameter configuration with $a_1 = 1, a_2 = 1.1, \delta_1 = \delta_2 = 0.15, \sigma_1 = \sigma_2 = 0.1, r = 0.03, b = 0.75, q_1 = 1,$ and $q_2 = 1.1$. This example is designed so that the upgraded technology is neutral in the sense that the optimal capital stock in the absence of the upgrading opportunity is $k_1^*(0) = 291.63$, and this coincides with the optimal capital stock associated with the upgraded technology $k_2^* = 291.63$. However, as Figure 2 shows, the optimal capital stock increases sharply for small positive values of λ associated with a long expected delay until the upgraded technology arrives. The curve describing the optimal capital stock reaches a maximum, after which it is a decreasing function. The optimal

capital stock approaches $k_2^* = 291.63$, as λ grows over all bounds, i.e. as the updated technology becomes available immediately.

This "overshooting" phenomenon can be explained as follows. The acquisition costs (q) are incurred instantaneously, whereas the productivity gain from one unit of capital is spread over a longer time horizon. Thus, the firm has an incentive to allocate the acquisition of capital to the phase when the old and less costly technology is available. This tendency is very strong if the expected delay until the new technology arrives is long, and hence the optimal capital stock is initially increasing. However, as the firm faces a shorter expected delay until it can benefit from the stream of increased productivity (i.e. when λ is increased), the optimal capital stock is decreasing as it gradually adjusts to the updated technology. Overall, this example illustrates that the role of investment costs is more significant than that of productivity gains as a determinant of the optimal policy of investment accumulation. For that reason the optimal policy of investment accumulation exhibits an "overshooting" phase.

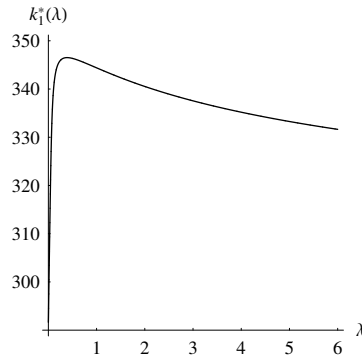


Figure 2: The optimal capital stock threshold $k_1^*(\lambda)$

In this respect our results seem to indicate that the role of investment costs is more significant (in percentage terms) than the role of productivity as a principal determinant of the optimal capital accumulation policy of a rationally managed firm. A natural explanation for such an observation is the fact that the acquisition costs of a new productive unit of capital are incurred instantaneously while the productivity of a unit of capital is spread over a longer time horizon and is, therefore, potentially productive under the more productive future technological generation. This observation naturally emphasizes the intertemporal trade-off between revenues and costs and in that respect it is in line with the real option literature on irreversible investment. Moreover, our results seem to indicate that although a cost-reducing technology generation increases the current

value of the firm by increasing the value of the future revenue flows it simultaneously increases the cost savings associated with the acquisition of the future productive capital stock. Since the latter of these effects dominates the former, we find that the net impact of a cost reducing technology is to decelerate investment in the current less efficient production technology.

3 Generalization: n Sequential Technology Generations

In this section we plan to extend the results of our previous section to include the possibility of n sequential technology generations affecting the investment decision of the firm. In order to accomplish this task, we now assume that the stochastic capital accumulation dynamics are

$$dK_t^I = -\delta_t K_t^I dt + \sigma_t K_t^I dW_t + dI_t, \quad K_0 = k, \quad (3.1)$$

where

$$\delta_t = \sum_{j=1}^{n-1} \delta_j \chi_{[\tau_{j-1}, \tau_j)}(t) + \delta_n \chi_{[\tau_{n-1}, \infty)}(t) \quad \text{and} \quad \sigma_t = \sum_{j=1}^{n-1} \sigma_j \chi_{[\tau_{j-1}, \tau_j)}(t) + \sigma_n \chi_{[\tau_{n-1}, \infty)}(t)$$

denote the percentage depreciation rate and the volatility coefficient of the capital stock K_t , respectively. Further, $\tau_0 = 0$, and $\{\tau_j\}_{j=1}^{n-1}$ is a sequence of exponentially distributed random arrival dates (conditional on the arrival date of the previous technology generation) characterizing the date at which the next upgraded technology generation is expected to arrive. For the sake of generality, we assume that the parameters $\lambda_j \in \mathbb{R}_+$, $j = 1, \dots, n-1$, of the exponential distributions need not to be identical but may vary as functions of the vintage $j \in \{1, \dots, n\}$ of the operating technology. Again, in the absence of interventions the capital accumulation dynamics constitute a coupled diffusion

$$K_t = \sum_{j=1}^{n-1} K_t^j \chi_{[\tau_{j-1}, \tau_j)}(t) + K_t^n \chi_{[\tau_{n-1}, \infty)}(t)$$

where the process K_t^i evolves on \mathbb{R}_+ according to the dynamics

$$dK_t^i = -\delta_i K_t^i dt + \sigma_i K_t^i dW_t, \quad i = 1, 2, \dots, n.$$

In accordance with the notation introduced in the previous section, we denote as

$$(R_r^i f)(k) = \mathbf{E}_k \int_0^\infty e^{-rs} f(X_s^i) ds$$

the expected cumulative present value of the cash flow $f(k)$ (whenever it exists) given that the firm operates the i th generation of the production technology.

We now plan to generalize the results of our previous section and determine the irreversible investment policy $I_t \in \Lambda$ for which the maximum

$$V(t, k) = \sup_{I \in \Lambda} \mathbf{E}_{(t, k)} \int_t^\infty e^{-rs} [\pi(s, K_s^I) ds - q_s dI_s], \quad \forall (t, k) \in \mathbb{R}_+^2 \quad (3.2)$$

is attained. In (3.2)

$$q_t = \sum_{j=1}^{n-1} q_j \chi_{[\tau_{j-1}, \tau_j)}(t) + q_n \chi_{[\tau_{n-1}, \infty)}(t)$$

denotes the unit cost of investment and

$$\pi(t, k) = \sum_{j=1}^{n-1} \pi_j(k) \chi_{[\tau_{j-1}, \tau_j)}(t) + \pi_n(k) \chi_{[\tau_{n-1}, \infty)}(t)$$

denotes the sequence of short-run revenue flows generated by the firm's current version of the production technology. We assume that the revenue flows $\pi_i(k)$ satisfy the Inada conditions stated in the previous section.

Following the analysis in the previous section, we now define the sequence $\{\theta_j(k)\}_{j=1}^n$ as

$$\begin{aligned} \theta_j(k) &= \pi_j(k) - (r + \delta_j)q_j k + \lambda_j(V_{j+1}(k) - q_j k), \quad j \leq n-1 \\ \theta_n(k) &= \pi_n(k) - (r + \delta_n)q_n k, \end{aligned}$$

where $V_j(k)$ denotes the value of the firm operating with the j th version of the production technology. Again, the principle of optimality implies that the optimal investment problem (3.2) has to be solved by applying backward recursion.

We first associate to each generation $j = 1, \dots, n$ the ordinary non-linear programming problem

$$\sup_{k \in \mathbb{R}_+} \left\{ k^{1-\varphi_j} (R_{r+\lambda_j}^j \theta_j)'(k) \right\}, \quad (3.3)$$

where $\varphi_j < 0$ denotes the negative and $\psi_j > 1$ the positive root of the characteristic equation $\sigma_j^2 a(a-1)/2 - \delta_j a - (r + \lambda_j) = 0$ with $\lambda_n = 0$. In accordance with the findings in our previous section, we observe that after the last technological version has arrived, that is, when $t \geq \tau_{n-1}$ the problem is completely analogous to problem (2.6) and, therefore, we find that

$$V_n(k) = \begin{cases} q_n k + (R_r^n \theta_n)(k) - \frac{1}{\varphi_n} k_n^* (R_r^n \theta_n)'(k_n^*) (k/k_n^*)^{\varphi_n} & k > k_n^* \\ q_n k + \theta_n(k_n^*)/r & k \leq k_n^* \end{cases} \quad (3.4)$$

where $k_n^* = \operatorname{argmax} \{k^{1-\varphi_n} (R_r^n \theta_n)'(k)\} \in (0, \tilde{k}_n)$ is the unique root of the ordinary first order condition $k_n^* (R_r^n \theta_n)''(k_n^*) = (\varphi_n - 1)(R_r^n \theta_n)'(k_n^*)$, and $\tilde{k}_n = \operatorname{argmax} \{\theta_n(k)\}$ is the unique root of

the first order condition $\theta'_n(\tilde{k}_n) = 0$. As in the previous section, the ordinary first order condition $k_n^*(R_r^n \theta_2)''(k_n^*) = (\varphi_n - 1)(R_r^n \theta_n)'(k_n^*)$ can be re-expressed as

$$\int_{k_n^*}^{\infty} y^{-\psi_n} \theta'_n(y) dy = 0. \quad (3.5)$$

The value function $V_n(k)$ is concave and, satisfies the inequalities $0 \leq V'_n(k) \leq q_n$. From these features we can conclude that $\theta_j(k)$ is strictly concave for all $j = 1, \dots, n-1$ and, therefore, that the mappings $\theta_j(k)$ are strictly concave for all $j = 1, \dots, n-1$. Moreover, we again find that there is a unique threshold \tilde{k}_j satisfying the first order condition $\theta'_j(\tilde{k}_j) = 0$ for all $j = 1, \dots, n-1$. Given these observations we find that there is a unique optimal capital stock threshold $k_j^* = \operatorname{argmax} \{k^{1-\varphi_j} (R_r^j \theta_j)'(k)\} \in (0, \tilde{k}_j)$ satisfying the optimality condition $k_j^* (R_{r+\lambda}^j \theta_j)''(k_j^*) = (\varphi_j - 1)(R_{r+\lambda}^j \theta_j)'(k_j^*)$, that is, the condition

$$\int_{k_j^*}^{\infty} y^{-\psi_j} \theta'_j(y) dy = 0.$$

In line with the findings of Corollary 2.6, we can draw the following conclusion.

Corollary 3.1. *If*

$$\int_{k_{j+1}^*}^{\infty} y^{-\psi_j} \theta'_j(y) dy \leq 0,$$

then $k_j^* \leq k_{j+1}^*$.

Proof. The proof is completely analogous to the proof of Corollary 2.6. \square

As one can conclude from Corollary 3.1, the sequence of optimal capital stock thresholds $\{k_j^*\}_{j=1}^n$ is non-decreasing, meaning that technological progress will stimulate investment, if

$$\int_{k_{j+1}^*}^{\infty} y^{-\psi_j} \theta'_j(y) dy \leq 0$$

for all $j = 1, \dots, n$. Moreover, the optimal capital accumulation policy is neutral with respect to technological change whenever

$$\int_{k_{j+1}^*}^{\infty} y^{-\psi_j} \theta'_j(y) dy = 0.$$

By comparing Corollary 3.1 with Corollary 2.6 we can conclude that our analysis can be generalized from a horizon of two successive technology generations to a horizon with an arbitrary, but finite, number of technology generations in a straightforward manner. Accordingly, the value of the optimal policy reads as

$$V_j(k) = \begin{cases} q_j k + (R_{r+\lambda_j}^k \theta_j)(k) - \frac{1}{\varphi_j} k_j^* (R_{r+\lambda_j}^j \theta_j)'(k_j^*) (k/k_j^*)^{\varphi_j} & k > k_j^* \\ q_j k + \theta_j(k_j^*) / (r + \lambda_j) & k \leq k_j^*. \end{cases} \quad (3.6)$$

Interestingly, we once again find that the marginal value of the firm operating with a version k of the production technology reads as

$$V_j'(k) = q_j + (R_{r+\lambda_j}^j \theta_j)'(k) - k^{\varphi_j-1} \sup_{y \leq k} \left[y^{1-\varphi_j} (R_{r+\lambda_j}^j \theta_j)'(y) \right].$$

This can be re-expressed as

$$V_j'(k) = q_j + (R_{r+\lambda_j}^j \theta_j)'(k) - \sup_{\tau} \mathbf{E}_k \left[e^{-(r+\lambda_j+\delta_j)\tau} (R_{r+\lambda_j}^j \theta_j)'(\hat{K}_{\tau}^j) \right],$$

where the process \hat{K}_t^j evolves according to the dynamics described by the stochastic differential equation

$$d\hat{K}_t^j = (\sigma_j^2 - \delta_j) \hat{K}_t^j dt + \sigma_j \hat{K}_t^j dW_t, \quad \hat{K}_0^j = k.$$

4 Concluding Remarks

In this study we have applied a real options perspective to develop a general characterization of the dynamics of the capital accumulation process in the presence of technological progress. We have delineated circumstances under which the present technology represents a compound real option, which incorporates as valuable embedded real options the opportunity of successive updating of this technology to superior future generations with a stochastic pattern of arrival timings. For example, with constant rates of depreciation and volatility we found that technological improvements in the form of cost reductions or revenue increases will induce a monotonic sequence of continuously increasing capital stock thresholds. We have also characterized how the extended option value created by technological progress generates modifications of the investment volumes in anticipation of future technology improvements. In particular, we have found that a shorter expected delay until a value-increasing technology is available induces a lower optimal capital stock. Intuitively, this captures the idea that the firm has a stronger incentive to slow down its investment under the present technology and wait for the arrival of the improved technology the shorter is the expected delay for this arrival.

In many contexts technology policy recommendations tend to emphasize the significance of supporting early technology versions with a potential of generating a future stream of successively improving technology generations. These days such recommendations are applied to, for example, the biotechnology industry. Our present model does not necessarily dispute such a policy recommendation. However, our model implies that many of the models applied to justify such a policy conclusion might exaggerate the significance of such policies as these models are

not formulated within the framework of a real options perspective. As our analysis makes clear, a rational evaluation of the embedded options created through technological progress will generate capital accumulation dynamics which will internalize the intertemporal spillovers between successive technology generations. Thus, technology policy interventions have to be justified by reference to other types of inefficiencies.

As always, a number of simplifications might limit the generality of the present analysis. Most importantly, our model is formulated within the framework of exogenous technological progress. Thus, our model does not explain technological progress and it cannot be used to explore, for example, the relationship between market structure and innovation. However, our model seems to be a useful building block for attempts to endogenize R&D investments. Likewise, the present model does not explore aspects of the strategic interaction between oligopolists engaged in time-based competition in the form of R&D races (see, for example, Weeds 2002 and Miltersen and Schwartz 2003). In fact, these are dimensions along which the present model could potentially be generalized.

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References

- [1] Alvarez, L. H. R. *A class of solvable singular stochastic control problems*, 1999, *Stochastics and Stochastics Reports*, **67**, 83–122.
- [2] Alvarez, L. H. R. and Stenbacka, R. *Adoption of Uncertain Multi-stage Technology Projects: A Real Options Approach*, 2001, *Journal of Mathematical Economics*, **35**, 71–97.
- [3] Alvarez, L. and R. Stenbacka *Optimal Risk Adoption: A Real Options Approach*, 2004, *Economic Theory*, **23**, 123–148.
- [4] Arrow, K. J. *Optimal capital policy with irreversible investment*, 1968, in *Value, Capital and Growth: papers in honour of Sir John Hicks*, ed. Wolfe, J. N., Chicago, 1–19.

- [5] Baldursson, F. M. and Karatzas, I. *Irreversible investment and industry equilibrium*, 1997, *Finance and Stochastics*, **1**, 69 – 89.
- [6] Baldwin, C. *Optimal sequential investment when capital is not readily reversible*, 1982, *Journal of Finance*, **37**, 763 – 782.
- [7] Borodin, A. and Salminen, P. *Handbook on Brownian motion - Facts and formulae*, 1996, Birkhauser, Basel.
- [8] Dixit, A. K. and Pindyck, R. S. *Investment under uncertainty*, 1994, Princeton UP, Princeton.
- [9] Dutta, P. *Optimal Management of an R&D Budget*, 1997, *Journal of Economic Dynamics and Control*, **21**, 575–602.
- [10] Harrison, J. M. *Brownian motion and stochastic Flow Systems*, 1985, Wiley, New York.
- [11] Jorgenson, D. *Capital theory and investment behavior*, 1963, *American Economic Review*, vol. 53, 247-259.
- [12] Kobila, T. Ø. *A class of solvable stochastic investment problems involving singular controls*, 1993, *Stochastics and Stochastics Reports*, vol 43, 29 - 63.
- [13] Merton, R. C. *An Asymptotic Theory of Growth Under Uncertainty*, 1975, *Review of Economic Studies*, **42**, 375–393.
- [14] Miltersen, K. R. and Schwartz, E. S. *R&D Investments with Competitive Interactions*, 2003, working paper, UCLA.
- [15] Nickell, S. J. *On the role of expectations in the pure theory of investment*, 1974, *Review of Economic Studies*, **41**, 1 – 19.
- [16] Øksendal, B. *Stochastic differential equations: An introduction with applications*, (Fifth Edition) 1998, Springer, Berlin.
- [17] Weeds, H. *Strategic Delay in a Real Options Model of R&D Competition*, 2002, *Review of Economic Studies*, **69**, 729–747.

A Proof of Lemma 2.1

Proof. As is known from the theory of linear diffusions, the expected cumulative present value of the flow $\theta_2(k)$ can be re-expressed as (cf. Kobilá 1993)

$$(R_r^2 \theta_2)(k) = \frac{2}{\sigma_2^2(\psi_2 - \varphi_2)} \left[k^{\varphi_2} \int_0^k y^{-1-\varphi_2} \theta_2(y) dy + k^{\psi_2} \int_k^\infty y^{-1-\psi_2} \theta_2(y) dy \right]. \quad (\text{A.1})$$

Consequently, we find that the non-linear programming problem (2.8) can be restated as

$$\sup_{k \in \mathbb{R}_+} \left\{ \frac{2}{\sigma_2^2(\psi_2 - \varphi_2)} \left[\varphi_2 \int_0^k y^{-1-\varphi_2} \theta_2(y) dy + \psi_2 k^{\psi_2 - \varphi_2} \int_k^\infty y^{-1-\psi_2} \theta_2(y) dy \right] \right\}. \quad (\text{A.2})$$

Define now the mapping $g : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$g(k) = \frac{2}{\sigma_2^2(\psi_2 - \varphi_2)} \left[\varphi_2 \int_0^k y^{-1-\varphi_2} \theta_2(y) dy + \psi_2 k^{\psi_2 - \varphi_2} \int_k^\infty y^{-1-\psi_2} \theta_2(y) dy \right].$$

Standard differentiation then yields that $g'(k) = 2\psi_2 k^{\psi_2 - \varphi_2 - 1} I(k) / \sigma_2^2$, where

$$I(k) = \int_k^\infty y^{-1-\psi_2} (\theta_2(y) - \theta_2(k)) dy. \quad (\text{A.3})$$

Since $\theta_2(k)$ is decreasing for $k \geq \tilde{k}_2$, we find that $I(k) \leq 0$ for all $k \in [\tilde{k}_2, \infty)$. Applying the mean value theorem for integrals then yields that for any $k \in (0, \tilde{k}_2)$ we have

$$I(k) = \frac{k^{-\psi_2}}{\psi_2} (\theta_2(\xi) - \theta_2(k)) + \int_{\tilde{k}_2}^\infty y^{-1-\psi_2} \theta_2(y) dy - \frac{\theta_2(\xi)}{\psi_2} \tilde{k}_2^{-\psi_2},$$

where $\xi \in (k, \tilde{k}_2)$. Letting k then tend to zero implies that $\lim_{k \downarrow 0} I(k) = \infty$ proving that $I(k)$ has a root $k_2^* \in (0, \tilde{k}_2)$. Since $I'(k) = -\theta_2'(k) k^{-\psi_2} / \psi_2 < 0$ for all $k \in (0, \tilde{k}_2)$ the uniqueness of the root k_2^* follows from the monotonicity of $I(k)$ on $(0, \tilde{k}_2)$. Consequently, we find that $g'(k) \gtrless 0$ when $k \gtrless k_2^*$. Moreover, standard integration by parts in (A.3) then results in

$$I(k) = \frac{1}{\psi_2} \int_k^\infty y^{-\psi_2} \theta_2'(y) dy,$$

completing the proof of our lemma. \square

B Proof of Theorem 2.2

Proof. Instead of considering the optimal investment problem (2.6) directly, we analyze the associated problem

$$F(k) = q_2 k + \sup_{I \in \Gamma} \mathbf{E}_k \int_0^\infty e^{-rs} \theta_2(K_s^I) ds, \quad (\text{B.1})$$

where Γ denotes the class of local time controls (i.e. the class of *local time pushes*, cf. Harrison 1985) of the type

$$I_t(y) = \begin{cases} (y - k)^+ & t = 0 \\ L_2(t, y) & t > 0, \end{cases} \quad (\text{B.2})$$

where $y \in \mathbb{R}_+$ and $L_2(t, y)$ denotes the local time of the capital stock process K_t^2 at the state $y \in \mathbb{R}_+$. Put somewhat differently, the class Γ consists of those investment policies which are such that they keep the capital stock associated with the upgraded technology above the threshold y . Since $\Gamma \subset \Lambda$, we obviously have that $V_2(k) \geq F(k)$ for all $x \in \mathbb{R}_+$. In order to prove the opposite inequality, we first observe that given an arbitrary investment policy $I_t^y \in \Gamma$, the resulting capital stock process constitutes a geometric Brownian motion reflected upwards at y implying that the standard transversality condition $\lim_{t \rightarrow \infty} \mathbf{E}_k [e^{-rt} K_t^I] = 0$ is met and, therefore, that for an arbitrary policy $I_t^y \in \Gamma$ the objective functional in (B.1) can be re-expressed as

$$F_y(k) = q_2 k + \mathbf{E}_k \int_0^\infty e^{-rs} [\pi_2(\hat{K}_s) - q_2(r + \delta_2)\hat{K}_s] ds, \quad (\text{B.3})$$

where \hat{K}_t denotes a geometric Brownian motion reflected at y . Denote now as

$$\mathcal{A}_i = \frac{1}{2} \sigma_i^2 k^2 \frac{d^2}{dk^2} - \delta_i k \frac{d}{dk}$$

the differential operator associated with the capital accumulation dynamics K_t^i , $i = 1, 2$. Since the value $F_y(k)$ satisfies on (y, ∞) the ordinary differential equation $(\mathcal{A}_2 F_y)(k) - r F_y(k) + \pi_2(k) = 0$ subject to the boundary condition $F_y'(y) = q_2$ and $F_y'(k) = q_2$ for all $k \in (0, y)$, we find that

$$F_y(k) = \begin{cases} q_2 k + (R_r^2 \theta_2)(k) - \frac{1}{\varphi_2} (R_r^2 \theta_2)'(y) y^{1-\varphi_2} k^{\varphi_2} & k > y \\ q_2 k + (R_r^2 \theta_2)(y) - \frac{1}{\varphi_2} (R_r^2 \theta_2)'(y) y & k \leq y. \end{cases}$$

It is then clear that the first order condition $k_2^* (R_r^2 \theta_2)''(k_2^*) = (\varphi_2 - 1) (R_r^2 \theta_2)'(k_2^*)$ guarantees that $F_{k_2^*}(k)$ is twice continuously differentiable on \mathbb{R}_+ . Moreover, since $\psi_2 \varphi_2 = -2r/\sigma_2^2$ we find that

$$(R_r^2 \theta_2)(k_2^*) - \frac{1}{\varphi_2} (R_r^2 \theta_2)'(k_2^*) k_2^* = \frac{\theta_2(k_2^*)}{r}$$

implying that $F_{k_2^*}(k)$ is equal to the proposed value function (2.10). Standard differentiation of $F_{k_2^*}(k)$ implies that on (k_2^*, ∞)

$$F_{k_2^*}'(k) = q_2 + k^{\varphi_2-1} \left[k^{1-\varphi_2} (R_r^2 \theta_2)'(k) - (R_r^2 \theta_2)'(k_2^*) k_2^{*1-\varphi_2} \right] \leq q_2$$

and, therefore, that $F_{k_2^*}'(k) \leq q_2$ for all $k \in \mathbb{R}_+$. Moreover, since

$$(\mathcal{A}_2 F_{k_2^*})(k) - r F_{k_2^*}(k) + \pi_2(k) = \theta_2(k) - \theta_2(k_2^*) < 0$$

for all $k \in (0, k_2^*)$, we find that $(\mathcal{A}_2 F_{k_2^*})(k) - rF_{k_2^*}(k) + \pi_2(k) \leq 0$ for all $k \in \mathbb{R}_+$ and, therefore, that $F_{k_2^*}(k)$ satisfies the sufficient variational inequalities

$$\min\{rF_{k_2^*}(k) - \pi_2(k) - (\mathcal{A}_2 F_{k_2^*})(k), q_2 - F'_{k_2^*}(k)\} = 0.$$

This result, in turn, implies that $F_{k_2^*}(k) \geq V_2(k)$ and, therefore that $F_{k_2^*}(k) = V_2(k)$.

Having established that $V_2(k) = F_{k_2^*}(k)$ it remains to prove that $V_2'(k) \geq 0$, that $V_2(k)$ is concave, and that $\lim_{k \rightarrow \infty} V_2'(k) = 0$. The monotonicity of $V_2(k)$ follows directly from the decomposition $(R_r^2 \theta_2)(k) = (R_r^2 \pi_2)(k) - q_2 k$ implying that on (k_2^*, ∞) we have

$$V_2'(k) = k^{\varphi_2 - 1} \left[k^{1 - \varphi_2} (R_r^2 \pi_2)'(k) - k_2^{*1 - \varphi_2} (R_r^2 \pi_2)'(k_2^*) \right] + q_2 (k/k_2^*)^{\varphi_2 - 1} \geq 0.$$

To prove the concavity of the value, we first notice that on (k_2^*, ∞)

$$V_2''(k) = k^{\varphi_2 - 2} \left[k^{2 - \varphi_2} (R_r^2 \theta_2)''(k) - k_2^{*2 - \varphi_2} (R_r^2 \theta_2)''(k_2^*) \right].$$

Consider now the functional $k^{2 - \varphi_2} (R_r^2 \theta_2)''(k)$. Standard differentiation then yields

$$\frac{d}{dk} \left[k^{2 - \varphi_2} (R_r^2 \theta_2)''(k) \right] = \frac{2}{\sigma_2^2} k^{\psi_2 - \varphi_2 - 1} \left[\psi_2 (\psi_2 - 1) \int_k^\infty y^{-1 - \psi_2} (\theta_2(y) - \theta_2(k)) dy - k^{1 - \psi_2} \theta_2'(k) \right].$$

The strict concavity of the mapping $\theta_2(k)$ then implies that $\theta_2(y) \leq \theta_2(k) + \theta_2'(k)(y - k)$ proving that $\frac{d}{dk} \left[k^{2 - \varphi_2} (R_r^2 \theta_2)''(k) \right] \leq 0$ and, therefore, that the value $V_2(k)$ is concave. To prove that $\lim_{k \rightarrow \infty} V_2'(k) = 0$, we first observe that

$$(R_r^2 \pi_2)'(k) = \mathbf{E}_k \int_0^\infty e^{-(r + \delta_2)s} \pi_2'(\hat{K}_s^2) ds,$$

where \hat{K}_t^2 is defined as in (2.13). The Inada condition $\lim_{k \rightarrow \infty} \pi_2'(k) = 0$ then implies that $\lim_{k \rightarrow \infty} (R_r^2 \pi_2)'(k) = 0$ since ∞ is a natural boundary for geometric Brownian motion. Thus, we find that

$$\lim_{k \rightarrow \infty} V_2'(k) = \lim_{k \rightarrow \infty} (R_r^2 \pi_2)'(k) - \lim_{k \rightarrow \infty} k_2^{*1 - \varphi_2} k^{\varphi_2 - 1} (R_r^2 \theta_2)'(k_2^*) = 0$$

completing the proof of our Theorem. \square

C Proof of Theorem 2.3

Proof. Denote as $\tilde{V}_2(k)$ the value of the firm and as \bar{k}_2^* the optimal investment threshold in the presence of the volatility coefficient $\tilde{\sigma}_2$ satisfying the inequality $\tilde{\sigma}_2 > \sigma_2$. Since $V_2(k)$ is concave, we find that

$$\frac{1}{2} \tilde{\sigma}_2^2 k^2 V_2''(k) - \delta_2 k V_2'(k) - r V_2(k) + \pi_2(k) \leq \frac{1}{2} (\tilde{\sigma}_2^2 - \sigma_2^2) k^2 V_2''(k) \leq 0$$

for all $k \in \mathbb{R}_+$. Moreover, since $V_2'(k) \leq q_2$ we find that $V_2(k) \geq \tilde{V}_2(k)$. Assume that $k < \min(k_2^*, \bar{k}_2^*)$. Then, the inequality

$$V_2(k) - \tilde{V}_2(k) = \frac{\theta_2(k_2^*) - \theta_2(\hat{k}_2^*)}{r} \geq 0$$

implies that $k_2^* \geq \hat{k}_2^*$ since the optimal investment threshold is attained on the set where $\theta_2(k)$ is increasing. Establishing that increased investment costs (i.e. an increase in q_2) decelerate investment by decreasing the optimal investment threshold and decrease the value of the optimal capital accumulation policy is completely analogous with the proof above.

Denote now as $\tilde{V}_2(k)$ the value of the firm and as \bar{k}_2^* the optimal investment threshold in the presence of the discount rate \tilde{r} satisfying the inequality $\tilde{r} > r$. Since $V_2'(k) \leq q_2$ and

$$\frac{1}{2}\sigma_2^2 k^2 V_2''(k) - \delta_2 k V_2'(k) - \tilde{r} V_2(k) + \pi_2(k) \leq -(\tilde{r} - r)V_2(k) \leq 0$$

for all $k \in \mathbb{R}_+$, we find again that $V_2(k) \geq \tilde{V}_2(k)$. In order to establish that increased discounting decreases the optimal boundary, we first consider the associated optimal stopping problem (2.12) characterizing the marginal value of firm. The strong Markov property of diffusion implies that (2.12) can be re-expressed as

$$V_2'(k) = \inf_{\tau} \mathbf{E}_k \left[\int_0^{\tau} e^{-(r+\delta_2)s} \pi_2'(\hat{K}_s^2) ds + e^{-(r+\delta_2)\tau} q_2 \right].$$

It is now clear that $\tilde{V}_2'(k)$ satisfies the variational inequalities $\tilde{V}_2'(k) \leq q_2$ and

$$\frac{1}{2}\sigma_2^2 k^2 \tilde{V}_2'''(k) + (\sigma_2^2 - \delta_2)k \tilde{V}_2''(k) - (\tilde{r} + \delta_2)\tilde{V}_2'(k) + \pi_2'(k) \geq 0$$

for all $k \in \mathbb{R}_+ \setminus \{\bar{k}_2^*\}$ and that $\tilde{V}_2'''(\bar{k}_2^* \pm) < \infty$. Moreover, since

$$\frac{1}{2}\sigma_2^2 k^2 \tilde{V}_2'''(k) + (\sigma_2^2 - \delta_2)k \tilde{V}_2''(k) - (r + \delta_2)\tilde{V}_2'(k) + \pi_2'(k) \geq (\tilde{r} - r)\tilde{V}_2'(k) \geq 0$$

for all $k \in \mathbb{R}_+ \setminus \{\bar{k}_2^*\}$, we find that

$$\mathbf{E}_k \left[e^{-(r+\delta_2)T_n} q_2 \right] \geq \mathbf{E}_k \left[e^{-(r+\delta_2)T_n} \tilde{V}_2'(\hat{K}_{T_n}^2) \right] \geq \tilde{V}_2'(k) - \mathbf{E}_k \int_0^{T_n} e^{-(r+\delta_2)s} \pi_2'(\hat{K}_s^2) ds,$$

where $T_n = \min(n, \inf\{t \geq 0 : \hat{K}_t^2 \geq n\}, \tau)$ is an almost surely finite stopping time. Reordering terms then yields that

$$\tilde{V}_2'(k) \leq \mathbf{E}_k \left[e^{-(r+\delta_2)T_n} q_2 + \int_0^{T_n} e^{-(r+\delta_2)s} \pi_2'(\hat{K}_s^2) ds \right].$$

Letting $n \rightarrow \infty$ and invoking monotone convergence then implies that

$$\tilde{V}_2'(k) \leq \mathbf{E}_k \left[e^{-(r+\delta_2)\tau} q_2 + \int_0^{\tau} e^{-(r+\delta_2)s} \pi_2'(\hat{K}_s^2) ds \right].$$

Since this inequality holds for any stopping strategy it must hold for the optimal as well and, therefore, $\tilde{V}'_2(k) \leq V'_2(k)$ for all $k \in \mathbb{R}_+$. That is, increased discounting decreases the marginal value of the firm. Denote now the continuation region where investing is suboptimal as $C_r = \{k \in \mathbb{R}_+ : V'_2(k) < q_2\} = (k_2^*, \infty)$ and $C_{\tilde{r}} = \{k \in \mathbb{R}_+ : \tilde{V}'_2(k) < q_2\} = (\bar{k}_2^*, \infty)$. If $k \in C_r$, then $\tilde{V}'_2(k) \leq V'_2(k) < q_2$ implies that $k \in C_{\tilde{r}}$ as well and, therefore, that $C_{\tilde{r}} \subset C_r$ thus demonstrating that increased discounting slows down investment. Establishing that increased depreciation decreases both the value and the optimal investment threshold and that an increase in q_2 increases the marginal value is completely analogous with the proof above. \square

D Proof of Theorem 2.7

Proof. As was established in Theorem 2.5 the value $V_1(k)$ satisfies the condition $(r+\lambda)V_1(k_1^*(\lambda)) = \pi_1(k_1^*(\lambda)) + \lambda V_2(k_1^*(\lambda)) - \delta_1 q_1 k_1^*(\lambda)$. Differentiating this identity and invoking the boundary condition $V'_1(k_1^*(\lambda)) = q_1$ now yields that $\theta'_1(k_1^*(\lambda))k_1^*(\lambda) = V_1(k_1^*(\lambda)) - V_2(k_1^*(\lambda))$ from which the alleged result characterizing the comparative static properties of the optimal threshold $k_1^*(\lambda)$ follows.

In order to establish the latter inequality, we first observe that for all $k \leq k_2^*$

$$V_2(k) = q_2 k + \frac{1}{r} \theta_2(k_2^*) = q_2 k + \frac{1}{r} [\pi_2(k_2^*) - (r + \delta) q_2 k_2^*].$$

Since k_2^* is attained on the set where $\theta_2(k)$ is increasing we observe that if $k_1^*(\lambda) < k_2^*$ then

$$V_2(k) \geq q_2 k + \frac{1}{r} [\pi_2(k_1^*(\lambda)) - (r + \delta) q_2 k_1^*(\lambda)].$$

Letting $k \uparrow k_1^*(\lambda)$ and inserting the resulting inequality to the definition of $k_1^*(\lambda)$ then yields the alleged result. \square

E Proof of Proposition 2.8

Proof. Define now the mapping $H : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$H(k) = \int_k^\infty y^{-\psi_1} \theta'_1(y) dy.$$

Since $\theta_1(k) = a_1 k^b - (r + \delta_1 + \lambda) q_1 k + \lambda V_2(k)$, we find that

$$H(k) = \left[a_1 + \frac{\lambda a_2}{(r + m_2(b))} \right] \frac{b k^{b-\psi_1}}{\psi_1 - b} - \frac{(r + \delta_1 + \lambda) q_1}{\psi_1 - 1} k^{1-\psi_1} - \frac{\lambda(1-b) q_2 k_2^{*1-\varphi_2}}{(b-\varphi_2)(\psi_1-\varphi_2)} k^{\varphi_2-\psi_1}$$

for all $k \geq k_2^*$ and that

$$H(k) = \frac{a_1 b}{\psi_1 - b} \left[k^{b-\psi_1} - k_2^{*b-\psi_1} \right] + \frac{(\lambda q_2 - (r + \delta_1 + \lambda) q_1)}{\psi_1 - 1} \left[k^{1-\psi_1} - k_2^{*1-\psi_1} \right] + H(k_2^*)$$

for all $k < k_2^*$. The alleged result then follows by letting $k \rightarrow k_2^*$. □